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Two Generalizations of Shannon's Inequality and Their Applications in Source Coding

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ABSTRACT
In the present paper Shannon's Inequality and its two generalizations are defined. Two new generalized mean code word lengths are introduced and their bounds in terms of the generalized measures of entropies are studied by applying the new two generalizations of Shannon's inequality thus obtained. Particular cases are also discussed with a list of references in the end.

KEYWORDS
Shannon's inequality; Codeword length; Source Coding; Holder's inequality; Kraft inequality

1. Introduction

Let $\Delta n = \{P = (p_1, p_2, \ldots\ldots p_n)\,; p_i \geq 0, \sum_{i=1}^{n} p_i = 1\}, \quad n \geq 2$ be a set of n-complete distributions defined on a random variable X. Then for $P \in \Delta n$, Shannon's entropy (1948) is

$$H(P) = -\sum_{i=1}^{n} p_i \log p_i \tag{1}$$

The measure (1) has been studied and generalized by many authors. It has found a wide applications in various disciplines of social, biological and physical sciences.

Harvda and Charvat (1967) characterized non- additive generalized Entropy of degree α as given below:

$$H^a(P) = \frac{1}{1-\alpha}\left(\sum_{i=1}^{n} p_i^a - 1\right), \quad \alpha > 0,\ \text{and}\ \alpha \neq 1. \tag{2}$$

Further, Sharma and Mittal (1975) also characterized the following non-additive

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entropy of order α and type β :

$$H_a^f(P) = \frac{1}{2^{1-\beta}-1}\left[\left(\sum_{i=1}^{n} p_i^a\right)^{\frac{\beta-1}{\alpha-1}} - 1\right], \alpha, \beta > 0. \alpha \neq \beta \text{ and } \alpha \neq 1 \tag{3}$$

Without loss of generality (3) can also be written as

$$H_a^\rho(P) = \frac{1}{1-\beta}\left[\left(\sum_{i=1}^{n} p_i^a\right)^{\frac{\beta-1}{\alpha-1}} - 1\right] \tag{4}$$

For $P, Q \in \Delta_n$, Kerridge (1961) introduced inaccuracy measure α defined as

$$H(P \mid Q) = -\sum_{i=1}^{n} p_i \log q_i \tag{5}$$

A relation between H(P) and $H(P \mid Q)$ is known as Shannon's inequality is given below:

$$H(P) \leq H(P \mid Q) \tag{6}$$

The Shannon's inequality given by (6) can be generalized in so many ways.

Using the method of Campbell (1965), Lubbe (1978) generalized (6) for the case of entropy of order α and type β. Thus, two new generalizations of (6) are defined and their applications in source coding are studied.

2. First Generalization of Shannon's Inequality

For $P, Q \in \Delta n$, we define a generalized measure of inaccuracy as given by

$$H^\alpha(P \mid Q) = \frac{1}{1-\alpha}\left[\left(\sum_{i=1}^{n} p_i q_i^{\alpha-1}\right)^{\alpha} - 1\right], \alpha > 0, \alpha \neq 1 \tag{7}$$

Since (7) reduces to (5) when $\alpha \to 1$, therefore, it is a generalized measure of Kerridge's inaccuracy. It may be noted that when $p_i = q_i$, (7) reduces to the generalized entropy of degree α as given below:

$$H^\alpha(P) = \frac{1}{1-\alpha}\left[\left(\sum_{i=1}^{n} p_i^a\right)^{\alpha} - 1\right], \alpha > 0, \alpha \neq 1 \tag{8}$$

which is different from (2). However, it reduces to (1), when $\alpha \to 1$. Hence (7) is a new generalized measure of entropy of degree α different from Harvard and Charvat's entropy.

Next we prove a theorem to establish an inequality between (2) and (7).

Theorem 2.1. *For $P, Q \in \Delta_n$, the following inequality between (2) and (7) holds:*

$$H^{a}(P) \leq H^{\omega}(P \mid Q) \tag{9}$$

Under the condition

$$\sum_{i=1}^{n} q_i^{\pi} \leq 1 \tag{10}$$

It may noted that the equality holds in (9) if $q_i = pi/\left(\sum_{i=1}^{n} p_i^{a}\right)^{\frac{1}{a}}, i = 1, 2, 3, \ldots\ldots n$

Proof. **Case (a)** When $0 < a < 1$. By Holder's inequality we know

$$\left(\sum_{i=1}^{n} x_i^{p}\right)^{\frac{1}{p}} \cdot \left(\sum_{i=1}^{n} y_i^{q}\right)^{\frac{1}{q}} \leq \sum_{i=1}^{n} x_i y_i, \tag{11}$$

Where all $x_i, y_i > 0, i = 1, 2, 3, 4, \ldots\ldots\ldots..n$ and $1/p + 1/q = 1$, such that either $(p \neq 0) < 1$, $q < 0$ or $q < 1(\neq 0), p < 0$. It may be noted that equality holds if and if there exists a positive constant c such that

$$x_i^{p} = c y_i^{q} \tag{12}$$

On substituting $p = \frac{\alpha-1}{\alpha}, x_i = p_i^{\overline{\frac{\alpha}{\alpha-1}}} \cdot q_i^{\alpha}; \quad q = 1-\alpha, y_i = p_i^{\alpha/1-\alpha}$ in (11), we get

$$\left(\sum p_i q_i^{\alpha-1}\right)^{\alpha/\alpha-1} \left(\sum_{i=1}^{n} p_i^{\alpha}\right)^{1/1-\alpha} \leq \sum p_i^{\alpha/\alpha-1} q_i^{\alpha} p_i^{\alpha/1-\alpha} = \sum_{i=1}^{n} q_i^{\alpha} \leq 1 \tag{13}$$

It implies

$$\left(\sum p_i q_i^{\alpha-1}\right)^{\alpha/\alpha-1} \leq \left(\sum_{i=1}^{n} p_i^{\alpha}\right)^{1/\alpha-1} \tag{14}$$

Since $\alpha < 1$, therefore, by raising power $\alpha - 1$ and subtracting 1 from both sides, we have

$$\left(\sum p_i q_i^{\alpha-1}\right)^{\alpha} - 1 \geq \left(\sum_{i=1}^{n} p_i^{\alpha} - 1\right)$$

or

$$\frac{1}{1-\alpha}\left[\left(\sum_{i=1}^{n} p_i q_i^{\alpha-1}\right)^{\alpha} - 1\right] \geq \frac{1}{1-\alpha}\left(\sum_{i=1}^{n} p_i^{\alpha} - 1\right)$$

or

$$H^{\alpha}(P) \leq H^{\alpha}(P \mid Q) \tag{15}$$

Case (b) When $\alpha > 1$. In this case (14) can be written as $\left(\Sigma p_i q_i^{\alpha-1}\right)^{\alpha} \leq \Sigma p_i^{\alpha}$ Subtracting 1 from both sides and multiplying $\frac{1}{1-\alpha}$ we have

$$\frac{1}{1-\alpha}\left[\left(\sum_{i=1}^{n} p_i q^{\alpha-1}\right)^{\alpha} - 1\right] \geq \frac{1}{1-\alpha}\left(\sum_{i=1}^{n} p_i^{\alpha} - 1\right)$$

or

$$H^{\alpha}(P) \leq H^{\alpha}(P \mid Q) \tag{16}$$

From (15) and (16) we conclude that the generalized inequality (9) holds. We see that (9) reduces to (6) when $\alpha \to 1$. □

3. Application of First Generalized Shannon's Inequality

Let a finite set of n input symbols with probabilities $p_1, p_2, \ldots\ldots\ldots, p_n$ be encoded in terms of code alphabets $\{a_1, a_2, \ldots\ldots, a_D\}$. Then there exists a uniquely decipherable code with lengths $l_1, l_2, ..., l_n$ (Refer Feinstein (1958)) iff

$$\sum_{i=1}^{n} D^{-i} \leq 1 \tag{17}$$

and

$$L = \sum_{i=1}^{n} p_i l_i \tag{18}$$

where (18) is the average code word length of the code. Further, it has been shown [refer to Feinstein (1958)] that

$$H(P) \leq L \tag{19}$$

with the inequality if and only if $l_i = -\log p_i, i = 1, 2, ..., n$. The inequality (17) is known as Kraft's inequality.

Next, we define a generalized mean code word length as given below:

$$L^{\alpha}(P) = \frac{1}{1-\alpha}\left[\left(\sum_{i=1}^{n} p_i D^{-\psi_{(1)}(-\alpha)}\right)^{\alpha} - 1\right], \text{ where } \alpha > 0, \alpha \neq 1 \tag{20}$$

Next we prove a theorem on bounds of (20) in terms of (8) applying the generalized inequality given by (9).

Theorem 3.1. *Let $l_i, t = 1, 2, 3, \ldots\ldots, n$ be lengths of code words, satisfying the following generalized Kraft inequality for uniquely decipherable code:*

$$\sum_{i=1}^{n} D^{-\alpha t_i} \leq 1, \quad \alpha > 0 \tag{21}$$

Then the following inequality holds:

$$H^{\alpha}(P) \leq L^{\alpha}(P) < D^{\alpha(1-\alpha)}H^{\alpha}(P) + (1-\alpha)^{-1} \cdot \left(D^{\alpha(1-\alpha)} - 1\right) \tag{22}$$

Proof. Choosing $q_i = D^{-t_i}$ in (2.3) for each i , we get

$$H^{\alpha}(P) \leq L^{\alpha}(P), \tag{23}$$

The equality holds iff $D^{-i_i} = \frac{p_i}{\left(\sum_{i=1}^{n_i p_i^2}\right)^{1/\alpha}}, \quad i = 1, 2, \ldots\ldots\ldots, n$

Let us choose l_i such that $-\log_D \left(\frac{p_i}{(\sum_{m=1}^{n} p_1^q)^{1/\alpha}}\right) \leq l_i < -\log\left(\frac{p_i}{(\sum_{i=1}^{n} p_i^q)^{1/\alpha}}\right) + 1$.
Then we have

$$D^{-t_i} > \frac{p_i}{\left(\sum_{i=1}^{n} p_i^i\right)^{1/\alpha_D}} \tag{24}$$

Here again two cases arise
Case (a) When $0 < \alpha < 1$, then raising power $\alpha - 1$ to both side of (24) we have $D^{t_i(1-\alpha)} < \frac{p_1^{\alpha-1}}{(\sum_{i=1}^{n} p_i^{\alpha})^{\alpha-1}\alpha_{D^{\alpha-1}}}$, since $_{1-\alpha}$ is negative.
It implies

$$\sum_{i=1}^{n} p_i D^{L(1-\alpha)} < \left(\sum_{i=1}^{n} p_i^{\alpha}\right)^{\frac{1}{\alpha}} D^{1-\alpha} \tag{25}$$

Raising power α and subtracting 1 from both sides, we have

$$\begin{aligned}\left(\sum_{i=1}^{n} p_i D^{i(1-\alpha)}\right)^{\alpha} - 1 < \sum_{i=1}^{n} p_i^{\alpha} D^{\alpha(1-\alpha)} - 1 &= \sum_{i=1}^{n} p_i^{\alpha} D^{\alpha(1-\alpha)} - D^{\alpha(1-\alpha)} + D^{\alpha(1-\alpha)} - 1 \\ &= D^{\alpha(1-\alpha)} \left[\sum_{i=1}^{n} p_i^{\alpha} - 1\right] + D^{\alpha(1-\alpha)} - 1\end{aligned}$$

Multiplying by $(1-\alpha)^{-1} > 0$ throughout, we get

$$L^{\alpha}(P) < D^{\alpha(1-\alpha)}H^{\alpha}(P) + (1-\alpha)^{-1}\left(D^{\alpha(1-\alpha)} - 1\right) \tag{26}$$

Case (b) can be obtained on the same lines. Hence Theorem 3.1 is proved.
Particular Case:

When $\alpha = 1$,(3.6) reduces to $-\sum_{i=1}^{n} p_i \log p_i \leq \sum_{i=1}^{n} p_i l_i < -\sum_{i=1}^{n} p_i \log p_i + 1$, which is well known Shannon's Noiseless Coding Theorem. □

4. Second Generalization of Shannon's Inequality

For $P, Q \in \Delta_n$, we define another generalized measure of inaccuracy as given by

$$H_\beta^\alpha(P \mid Q) = \frac{1}{1-\beta}\left[\left(\sum_{i=1}^{n} p_i^{\frac{a^2-\alpha+1}{\alpha}} \frac{q}{1}_{\frac{\alpha-1}{\alpha}}\right)^{\left(\frac{\rho-1}{a-1}\right)\alpha} - 1\right], \alpha > 0, \alpha \neq \beta, \beta > 1. \tag{27}$$

When $\beta \to 1$, (27) reduces to

$$H_a(P \mid Q) = \frac{\alpha}{1-\alpha} \log\left(\sum_{i=1}^{n} p_i^{\frac{\alpha^2-\alpha+1}{\alpha}} q_i^{\frac{\alpha-1}{\alpha}}\right). \tag{28}$$

Since (28) reduces to (5), when $\alpha \to 1$, therefore, (28) is a new generalized measure of inaccuracy of order α. Thus, we call (27) as the generalized measure of inaccuracy of order α and degree β.

It is interesting to note that when $p_i = q_i$ for each i and $\beta \to \alpha$, (27) reduces to

$$H_\alpha(P) = \frac{1}{1-\alpha}\left[\left(\sum_{i=1}^{n} p_i^\alpha\right)^\alpha - 1\right], \tag{29}$$

which is (8) and further it reduces to (1), when $\alpha \to 1$.

Theorem 4.1. *For $P, Q \in \Delta_n$, the following holds*

$$H_\beta^\alpha(P) \leq H_\beta^\alpha(P \mid Q) \tag{30}$$

under the condition

$$\sum_{i=1}^{n} p_i^{\alpha-1} q_i \leq 1, \ \textit{when } a > 0 \tag{31}$$

and equality holds if $q_i = \frac{p_i}{\sum_{i=1}^{n_1^2} p_i^2}, i = 1, 2, \ldots\ldots\ldots n$, *where* $H_a^\rho(P)$ *and* $H_a^\rho(P \mid Q)$ *are given by (4) and (27) respectively.*

Proof. Here also two cases arise

(a) $0 < \alpha < 1, \beta > 1$ (b) $\alpha \geq 1, \beta > 1$

Case(a). When $0 < \alpha < 1, \beta > 1$, then by Holder's inequality

$$\left(\sum_{i=1}^{n} x_i^p\right)^{1/p} \cdot (y_i^q)^{1/q} \leq \sum_{i=1}^{n} x_i y_i \tag{32}$$

For all $x_i, y_i > 0, t = 1, 2, \ldots\ldots\ldots n$ and $\frac{1}{p} + \frac{1}{q} = 1$, Such that either $(p \neq 0) < 1$ and $q < 0$ or $q(\neq 0) < 1$ and $p < 0$. It may be seen that equality holds if and only if there exists a positive constant c such that

$$x_i^p = c y_i^p$$

On substituting $p = \frac{\alpha-1}{\alpha}, x_i = p_i^{\frac{\alpha^2-\alpha+1}{\alpha-1}} q_i, \quad q = 1-\alpha$ and $y_i = p_i^{\alpha/1-\alpha}$ in (32), we get

$$\left(\sum_{i=1}^{n} p_i^{\frac{\alpha^2-\alpha+1}{\alpha}} q_i^{\frac{\alpha-1}{\alpha}}\right)^{\alpha/\alpha-1} \cdot \left(\sum_{i=1}^{n} p_i^{\alpha}\right)^{1/1-\alpha} \leq \sum_{i=1}^{n} p_i^{\alpha-1} q, \quad \alpha > 0, \alpha \neq 1$$

Using the condition (31), we have

$$\left(\sum_{i=1}^{n} p_i^{\frac{\alpha^2-\alpha+1}{\alpha}} q_i^{\frac{\alpha-1}{\alpha}}\right)^{\alpha/\alpha-1} \cdot \left(\sum_{i=1}^{n} p_i^{\alpha}\right)^{1/1-\alpha \leq 1}$$

or

$$\left(\sum_{i=1}^{n} p_i^{\frac{\alpha^2-\alpha+1}{\alpha}} q_i^{\frac{\alpha-1}{\alpha}}\right)^{\alpha/\alpha-1} \leq \left(\sum_{i=1}^{n} p_i^{\alpha}\right)^{1/\alpha-1}$$

Raising power $\alpha - 1$ to both sides, we get

$$\left(\sum_{i=1}^{n} p_i^{\frac{\alpha^2-\alpha+1}{\alpha}} q_i^{\frac{\alpha-1}{\alpha}}\right)^{\alpha} \geq \sum_{i=1}^{n} p_i^{\alpha}, \tag{33}$$

since $\alpha < 1$ is negative. Raising power $\frac{\rho-1}{\alpha-1} < 0$ and subtracting 1 from both sides, we have

$$\left(\sum_{i=1}^{n} p_i^{\frac{\alpha^2-\alpha+1}{\alpha}} q_i^{\frac{\alpha-1}{\alpha}}\right)^{\left(\frac{\rho-1}{\alpha-1}\right)\alpha} - 1 \leq \left(\sum_{i=1}^{n} p_i^{\alpha}\right)^{\frac{\rho-1}{\alpha-1}} - 1 \tag{34}$$

Multiplying both sides by $(1-\beta)^{-1} < 0$, we get

$H_\beta^\alpha(P) \leq H_\beta^\alpha(P/Q)$, which is (30).

Case (b) when $\alpha \geq 1, \beta > 1$, then substituting

$p = \frac{\alpha-1}{\alpha}, \quad x_i = p_i^{\frac{\alpha^2-\alpha+1}{\alpha-1}} q_i, \quad q = 1-\alpha$ and $y_i = p_i^{\alpha/1-\alpha}$ in (32) we get

$$\left(\sum_{i=1}^{n} p_i^{\frac{\alpha^2-\alpha+1}{\alpha}} q_i^{\frac{\alpha-1}{\alpha}}\right)^{\alpha/\alpha-1} \cdot \left(\sum_{i=1}^{n} p_i^{\alpha}\right)^{1/1-\alpha} \leq \sum_{i=1}^{n} p_i^{\alpha-1} q_i \leq 1$$

It implies $\left(\sum_{i=1}^{n} p_i^{\frac{\alpha^2-\alpha+1}{\alpha}} q_i^{\frac{\alpha-1}{\alpha}}\right)^{\alpha/\alpha-1} \leq \left(\sum_{i=1}^{n} p_i^{\alpha}\right)^{1/\alpha-1}$

Raising power $\alpha - 1$ to both sides, we get

$$\left(\sum_{i=1}^{n} p_i^{\frac{a^2-\alpha+1}{\alpha}} q_i^{\frac{\alpha-1}{\alpha}}\right)^{\alpha} \leq \sum_{i=1}^{n} p_i^{\alpha}, \text{ since } \alpha \geq 1 \tag{35}$$

Raising power $\frac{\beta-1}{\alpha-1} > 0$ to both sides and subtracting 1 from both sides, we have

$$\left(\sum_{i=1}^{n} p_i^{\frac{\alpha^2-\alpha+1}{\alpha}} q_i^{\frac{\alpha-1}{\alpha}}\right)^{\left(\frac{\beta-1}{\alpha-1}\right)\alpha} - 1 \leq \left(\sum_{i=1}^{n} p_i^{\alpha}\right)^{\frac{\beta-1}{\alpha-1}} - 1.$$

Multiplying both sides by $(1-\beta)^{-1} < 0$, we get (30). Hence the theorem is proved for both cases. □

5. Application of Second Generalized Shannon's Inequality in Source Coding

In this section we study an application of Theorem 4.1 in source coding. Let us define a generalized mean code word length as given below:

$$L_{\alpha}^{\beta}(P) = \frac{1}{1-\beta}\left[\left(\sum_{i=1}^{n} p_i^{\frac{\alpha^3-\alpha+1}{\alpha}} D^{t\left(\frac{\mathrm{l}-\alpha}{\alpha}\right)}\right)^{\left(\frac{\beta-1}{\alpha-1}\right)a} - 1\right], \text{ where } \alpha, \beta > 0, \alpha \neq \beta, \alpha \neq 1 \neq \beta \tag{36}$$

Next we prove a theorem on the bounds of (36) in terms of (27) using the generalized inequality (30).

Theorem 5.1. *Let $l_i, i = 1, 2, 3 \ldots\ldots n$ be the lengths code of words satisfying the following generalized Kraft inequality:*

$$\sum_{i=1}^{n} p_i^{\alpha-1} \cdot D^{-t_i} \leq 1, \quad \textit{when} \quad \alpha > 1 \quad \textit{and} \quad \alpha \neq 1 \tag{37}$$

Then the following condition holds

$$H_{\beta}^{\alpha}(P) \leq L_{\beta}^{\alpha}(P) < D^{1-\beta} H_{\beta}^{\alpha}(P) + (1-\beta)^{-1} \cdot \left(D^{1-\beta} - 1\right), \tag{38}$$

where $H_{\beta}^{\alpha}(P)$ and $L_{\beta}^{\alpha}(P)$ are given by (4) and (36) respectively.

Proof. Substituting $q_i = D^{-l_i}$ in (30), we have

$H_{\beta}^{\alpha}(P) \leq \frac{1}{1-\beta}\left[\left(\sum_{i=1}^{n} p_i^{\frac{\alpha^2-\alpha+1}{\alpha}} D^{-li\left(\frac{1-\alpha}{\alpha}\right)}\right)^{\left(\frac{\beta-t}{\alpha-1}\right)\alpha} - 1\right]$. It implies

$$H_{\alpha}^{\beta}(P) \leq L_{\alpha}^{\beta}(P), \tag{39}$$

Thus, the first part of (38) is proved.

The inequality holds if $D^{t_i} = \frac{p_i}{\sum_{i=1}^{n} p_i^{\alpha}}, i = 1, 2, 3, \ldots\ldots\ldots, n$ and

$$l_i = -\log_D p_i + \log_D \left[\sum_{i=1}^{n} p_i^{\alpha}\right]; \quad i = 1, 2, 3, \ldots\ldots..n \tag{40}$$

Next we choose l_i such that

$$-\log_D \frac{p_i}{\sum_{i=1}^{n} p_i^{\alpha}} \leq l_i < -\log_D \frac{p_i}{\sum_{i=1}^{n} p_i^{\alpha}} + 1$$

and prove

$$D^{-l_i} > \frac{p_i}{\sum_{i=1}^{n} p_i^{\alpha^{\alpha} D}} \tag{41}$$

Here two cases arise: (a) $0 < \alpha < 1, \beta > 1$ (b) $\alpha > 1, \beta > 1$

Case (a) When $0 < \alpha < 1$, and $\beta > 1$, then raising power $\frac{\alpha-1}{\alpha}$ to both sides of (41), we get

$$D^{-t_i\left(\frac{1-\alpha}{\alpha}\right)} < \frac{p_i^{\frac{\alpha-1}{\alpha}}}{\left(\sum_{i=1}^{n} p_i^{\alpha}\right)^{\alpha-1/\alpha}}$$

Multiplying by $p_i^{\frac{\alpha^2-\alpha+1}{\alpha}}$ both sides and taking summations over i, we have

$$\sum_{i=1}^{n} p_i^{\frac{\alpha^2-\alpha+1}{\alpha}} D^{t\left(\frac{1-\alpha}{\alpha}\right)} < \frac{\sum_{i=1}^{n} p_i^{\alpha}}{\left(\sum_{i=1}^{n} p_i^{\alpha}\right)^{\frac{\alpha-1}{\alpha}} D^{\left(\frac{\alpha-1}{\alpha}\right)}}$$

And that implies $\sum_{i=1}^{n} p_i^{\frac{\alpha^2-\alpha+1}{\alpha}} D^{l\left(\frac{i-\alpha}{\alpha}\right)} < \left(\sum_{i=1}^{n} p_i^{\alpha}\right)^{1/\alpha} D^{\left(\frac{1-\alpha}{\alpha}\right)}$

Raising power $\alpha\left(\frac{\beta-1}{\alpha-1}\right) < 0$ and subtracting 1 from both sides, we get

$$\begin{aligned}
\left(\sum_{i=1}^{n} p_i^{\frac{\alpha^2-\alpha+1}{\alpha}} D^{l^l\left(\frac{1-\alpha}{\alpha}\right)}\right)^{\left(\frac{\beta-1}{\alpha-1}\right)\alpha} - 1 &> \left(\sum_{i=1}^{n} p_i^{\alpha}\right)^{\frac{\beta-1}{\alpha-1}} \cdot D^{1-\beta} - 1 \\
&= D^{1-\beta}\left[\left(\sum_{i=1}^{n} p_i^{\alpha}\right)^{\frac{\beta-1}{\alpha-1}} - 1\right] + D^{1-\beta} - 1 \qquad (42)
\end{aligned}$$

Multiplying both sides by $(1-\beta)^{-1} < 0$, we get

$$L^{\alpha}_{\beta}(P) < D^{1-\beta} \cdot H^{\alpha}_{\beta}(P) + (1-\beta)^{-1}\left(D^{1-\beta} - 1\right) \tag{43}$$

which is second part of (38). Hence Theorem (36) is proved.

Case (b) can be obtained on the same lines.

Particular Cases

(i) If $\beta = \alpha$, then (38) reduces to

$$H^{\alpha}(P) \leq L^{\alpha}(P) < D^{1-\alpha} H^{\alpha}(P) + (1-\beta)^{-1} \cdot \left(D^{1-\alpha} - 1\right), \tag{44}$$

where $H^{\alpha}(P) = (1-\alpha)^{-1}\left[\sum_{i=1}^{n} p_i^{\alpha} - 1\right], \alpha > 0, \alpha \neq 1$, is Harvada and Charvat (1967) entropy and

$$L^{\alpha}(P) = (1-\alpha)^{-1}\left[\left(\sum_{i=1}^{n} p_i^{\frac{\alpha^2-\alpha+1}{\alpha}} D^{-l_i\left(\frac{\alpha-1}{\alpha}\right)}\right)^{\alpha} - 1\right], \alpha > 0, \alpha \neq 1$$

is new code word length.

(ii) If $\beta \to 1$, then (38) becomes

$$H_{\alpha}(P) \leq L^{\alpha}(P) < H_{\alpha}(P) + \log D,$$

where $H_{\alpha}(P) = \frac{1}{1-\alpha} \log_D \sum_{i=1}^{n} p_i^{\alpha}, \alpha(\neq 1) > 0$, is Renyi's (1961) entropy and

$$L_{\alpha}(P) = \frac{\alpha}{1-\alpha} \log_D \sum_{i=1}^{n} p_i^{\frac{\alpha^2-\alpha+1}{\alpha}} D^{-li\left(\frac{\alpha-1}{\alpha}\right)}, \alpha(\neq 1) > 0.$$

is another new mean codeword length. $\square$

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